

Coinductive Proof Principles for Stochastic Processes

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Abstract

We give an explicit coinduction principle for recursively-defined stochastic processes. The principle applies to any closed property, not just equality, and works even when solutions are not unique. We illustrate the use of the rule in deriving properties of a simple coin-flip process.

1 Introduction

Coinduction has been shown to be a useful tool in functional programming. Streams, automata, concurrent and stochastic processes, and recursive types have been successfully analyzed using coinductive methods; see [1, 7, 3, 9, 5] and references therein.

Most approaches emphasize the relationship between coinduction and bisimulation. In Rutten's treatment [9] (see also [5, 1]), the coinduction principle states that under certain conditions, two bisimilar processes must be equal. For example, to prove the equality of infinite streams $\sigma = \text{merge}(\text{split}(\sigma))$, where merge and split satisfy the familiar coinductive definitions

$$\begin{aligned}\text{merge}(a :: \sigma, \tau) &= a :: \text{merge}(\tau, \sigma) \\ \#1(\text{split}(a :: b :: \rho)) &= a :: \#1(\text{split}(\rho)) \\ \#2(\text{split}(a :: b :: \rho)) &= b :: \#2(\text{split}(\rho)),\end{aligned}$$

it suffices to show that the two streams are bisimilar. An alternative view is that certain systems of recursive equations over a certain algebraic structure have unique solutions. Desharnais et al. [3, 7] study bisimulation in a probabilistic context. They are primarily interested in the approximation of one process with another. Again, they focus on bisimulation, but do not formulate an explicit coinduction rule.

The coinduction principle can be generalized to other properties besides equations and to situations in which the solutions are not unique. In this paper we introduce such a

generalization and illustrate its use with an extended example.

2 An Example

Consider the following procedure for simulating a coin of arbitrary real bias q , $0 \leq q \leq 1$, with a coin of arbitrary real bias p , $0 < p < 1$. We assume unit-time exact arithmetic on real numbers.

```
boolean qflip(q) {  
  if (p < q) {  
    if (pflip()) return true;  
    else return qflip(1-(1-q)/(1-p));  
  } else {  
    if (pflip()) return qflip(q/p);  
    else return false;  
  }  
}
```

Intuitively, if $p < q$ and the bias- p coin flip returns heads (true), then we halt and output heads; this gives a fraction p/q of the desired probability q of heads of the simulated bias- q coin. If the bias- p coin returns tails, we rescale the problem appropriately and call qflip tail-recursively. Similarly, if $p \geq q$ and the bias- p coin returns tails, then we halt and output tails, and if not, we rescale appropriately and call qflip tail-recursively.

On any input $0 \leq q \leq 1$, the probability of halting is 1, since the procedure halts with probability at least $\min(p, 1-p)$ in each iteration. The probability that qflip halts and returns heads on input q exists and satisfies the recurrence

$$H(q) = \begin{cases} p \cdot H(\frac{q}{p}), & \text{if } q \leq p, \\ p + (1-p) \cdot H(1 - \frac{1-q}{1-p}), & \text{if } q > p. \end{cases} \quad (1)$$

Now $H^*(q) = q$ is a solution to this recurrence, as can be seen by direct substitution. There are uncountably many other solutions as well, but these are all unbounded. Since

H^* is the unique bounded solution, it must give the probability of heads.

We can do the same for the expected running time. Let us measure the expected number of calls to `pflip` on input q . The expectation exists and is uniformly bounded on the unit interval by $1/\min(p, 1-p)$, the expected running time of a Bernoulli (coin-flip) process with success probability $\min(p, 1-p)$. From the program, we obtain the recurrence

$$\begin{aligned} E_0(q) &= \begin{cases} (1-p) \cdot 1 + p \cdot (1 + E_0(\frac{q}{p})), & \text{if } q \leq p, \\ p \cdot 1 + (1-p) \cdot (1 + E_0(1 - \frac{1-q}{1-p})), & \text{if } q > p \end{cases} \\ &= \begin{cases} 1 + p \cdot E_0(\frac{q}{p}), & \text{if } q \leq p, \\ 1 + (1-p) \cdot E_0(1 - \frac{1-q}{1-p}), & \text{if } q > p. \end{cases} \end{aligned}$$

The unique bounded solution to this recurrence is

$$E_0^*(q) = \frac{q}{p} + \frac{1-q}{1-p}. \quad (2)$$

That it is a solution can be ascertained by direct substitution; uniqueness requires a further argument. As above, there are uncountably many unbounded solutions, but since E_0^* is the unique bounded solution, it must give the expected running times.

So far there is nothing that cannot be handled with Rutten or Desharnais et al. approach. However, the situation gets more interesting when we observe that slight modifications of the algorithm lead to noncontinuous fractal solutions with no simple characterizations like (2). The fractal behavior of stochastic processes has been previously observed in [6].

Assume $p \leq 1-p$. Say we want to save time by taking off a larger fraction $1-p$ of the remaining “heads” weight when $q > 1-p$. In that case, we will halt and report heads if `pflip` gives tails.

```
boolean qflip(q) {
  if (1-p < q) {
    if (pflip()) return qflip(1-(1-q)/p);
    else return true;
  } else if (p < q) {
    if (pflip()) return true;
    else return qflip(1-(1-q)/(1-p));
  } else {
    if (pflip()) return qflip(q/p);
    else return false;
  }
}
```

The recurrence for the expected running time is

$$E_1(q) = 1 + r(q)E_1(f_1(q)), \quad (3)$$

where

$$\begin{aligned} f_1(q) &= \begin{cases} \frac{q}{p}, & \text{if } q \leq p \\ 1 - \frac{1-q}{1-p}, & \text{if } p < q \leq 1-p \\ 1 - \frac{1-q}{p}, & \text{if } 1-p < q \end{cases} \\ r(q) &= \begin{cases} 1-p, & \text{if } p < q \leq 1-p \\ p, & \text{otherwise.} \end{cases} \end{aligned}$$

Again, there is a unique bounded solution E_1^* , but there is no longer a nice algebraic characterization like (2). The solution for $p = 1/4$ is the noncontinuous fractal shown in Fig. 1. The fractal is shown compared to the line E_0^* . The

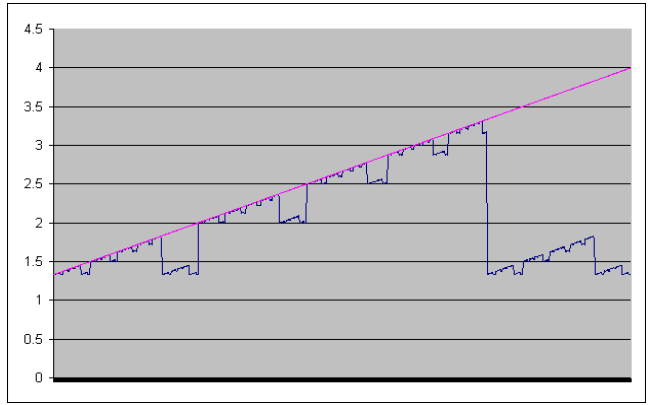


Figure 1. Fractal solution of (3)

large discontinuity at $q = 1-p = 3/4$ is due to the modification of the algorithm for $q > 1-p$, and this discontinuity is propagated everywhere by the recurrence.

Fig. 1 and intuition dictate that $E_1^* \leq E_0^*$, but how do we prove this? Not by induction, because there is no basis. An analytic argument involving convergence of sequences would be one possibility. However, there is a simpler alternative. It will follow from our coinductive proof principle that to conclude $E_1^* \leq E_0^*$, it suffices to show that $\tau(E)(q) \leq E_0^*(q)$ whenever $E(f_1(q)) \leq E_0^*(f_1(q))$, where τ is a suitably defined operator representing the unwinding of the recurrence. This property is easily checked, and no analysis is necessary.

We can modify the algorithm further to achieve more savings. If $1/2 < q \leq 1-p$, it would seem to our advantage to remove p from the tail probability of q rather than from the head probability. Although the weight removed in both cases is the same, savings for q in this region are realized in the next step.

```

boolean qflip(q) {
  if (1-p < q) {
    if (pflip()) return qflip(1-(1-q)/p);
    else return true;
  } else if (.5 < q) {
    if (pflip()) return false;
    else return qflip(q/(1-p));
  } else if (p < q) {
    if (pflip()) return true;
    else return qflip(1-(1-q)/(1-p));
  } else {
    if (pflip()) return qflip(q/p);
    else return false;
  }
}

```

The recurrence is

$$E_2(q) = 1 + r(q)E_2(f_2(q)) \quad (4)$$

with

$$f_2(q) = \begin{cases} \frac{q}{p}, & \text{if } q \leq p, \\ 1 - \frac{1-q}{1-p} & \text{if } p < q \leq 1/2, \\ \frac{q}{1-p} & \text{if } 1/2 < q \leq 1-p, \\ 1 - \frac{1-q}{p} & \text{if } 1-p < q, \end{cases}$$

and $r(q)$ as above. The symmetric fractal solution E_2^* is shown in Fig. 2. Intuition seems to say that this solution

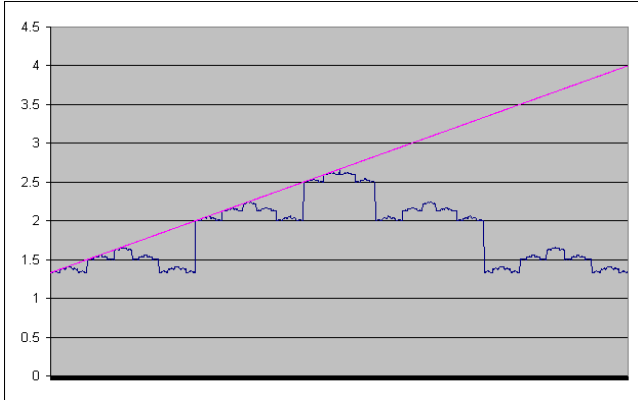


Figure 2. Fractal solution of (4)

should be better, but this is not always the case: for $p = 1/4$, there are countably many minute intervals on which $E_2^*(q) > E_1^*(q)$.

One might ask whether there a slight modification of E_2^* that is everywhere better than E_1^* . The answer is yes: take the breakpoint not at $1/2$, but at $c = \max((1-p)^2, 1 - (1-p)^2)$, provided $p \leq (1-p)^2$. For $p = 1/4$, this

would give $c = 9/16$. Now the recurrence is $E_3(q) = 1 + r(q)E_3(f_3(q))$ with

$$f_3(q) = \begin{cases} \frac{q}{p}, & \text{if } q \leq p, \\ 1 - \frac{1-q}{1-p} & \text{if } p < q \leq c, \\ \frac{q}{1-p} & \text{if } c < q \leq 1-p, \\ 1 - \frac{1-q}{p} & \text{if } 1-p < q. \end{cases} \quad (5)$$

Now we wish to show that $E_3^* \leq E_1^*$ on the whole unit interval. Note that we are comparing two nowhere-differentiable functions¹; we have no nice algebraic description of them save as solutions of the recurrences $E_i(q) = 1 + r(q)E_i(f_i(q))$. However, we can prove the desired inequality purely logically using the coinductive principle below, without recourse to analysis. We outline a proof below, after we have stated and proved the validity of the principle.

3 Statement and Proof of the Coinduction Principle

Here is a statement and proof of the coinduction principle. See [4] for the necessary background. Let B be a Banach space (complete normed linear space) over \mathbb{C} and let R be a bounded linear operator on B such that $I - R$ is invertible; that is, $1 \notin \sigma(R)$, where $\sigma(R)$ is the spectrum of R . Let $a \in B$. Since $I - R$ is invertible, there is a unique solution e^* to $e = a + Re$ given by $e^* = (I - R)^{-1}a$.

Theorem 3.1 *Consider the affine operator $\tau(e) = a + Re$, where R has spectral radius < 1 . Let $\varphi \subseteq B$ be a closed nonempty region preserved by τ . Then $e^* \in \varphi$.*

Proof. The spectral radius of R is

$$\sup_{\lambda \in \sigma(R)} |\lambda| = \inf_n \sqrt[n]{|R^n|}.$$

If this quantity is < 1 , then there exists n such that $|R^n| < 1$, thus $\sum_n R^n$ converges and equals $(I - R)^{-1}$. One can show that

$$|\tau^{m+k}(e_0) - \tau^m(e_0)| \leq \frac{|R^m|}{1 - |R|} |\tau(e_0) - e_0|,$$

thus the sequence $\tau^n(e_0)$ is a Cauchy sequence. Since B is a complete metric space, the sequence converges, and from the continuity of τ it follows that its limit is a fixpoint of τ , therefore must be e^* , the unique bounded solution of $e = a + Re$. If $e_0 \in \varphi$, then $\tau^n(e_0) \in \varphi$ for all n since τ preserves φ , and $e^* \in \varphi$ since φ is closed. \square

¹Hermite and Poincaré eschewed such functions, calling them a “dreadful plague”. Poincaré wrote: “Yesterday, if a new function was invented, it was to serve some purpose; today, they are invented only to debunk the arguments of our predecessors, and they will never have any other use.”

This translates to the following coinduction principle.

Theorem 3.2 *Let τ be as in Theorem 3.1. Let φ be a closed property. The following proof rule is valid:*

$$\frac{\exists e \varphi(e) \quad \forall e \varphi(e) \Rightarrow \varphi(\tau(e))}{\varphi(e^*)}. \quad (6)$$

More generally, for any $n \geq 1$,

$$\frac{\exists e \varphi(e) \quad \forall e \varphi(e) \Rightarrow \varphi(\tau^n(e))}{\varphi(e^*)}. \quad (7)$$

Proof. The rule (6) is just a restatement of Theorem 3.1. The rule (7) follows by applying (6) to the closed property $\psi(e) = \bigvee_{i=0}^{n-1} \varphi(\tau^i(e))$. This is a closed property because τ is continuous on B . \square

For example, to show that $E_1^* \leq E_0^*$ using the rule (6), we take B to be the space of bounded complex-valued functions on the unit interval, $a = \lambda x.1$, $R : B \rightarrow B$ the bounded linear operator $RE = \lambda q.r(q)E(f_1(q))$ with spectral radius $1 - p$, $\varphi(E)$ the closed property

$$\forall q \ E(q) \leq \frac{q}{p} + \frac{1-q}{1-p},$$

and

$$\tau(E) = \lambda q.(1 + r(q)E(f_1(q))),$$

where

$$f_1(q) = \begin{cases} \frac{q}{p}, & \text{if } q \leq p \\ 1 - \frac{1-q}{1-p}, & \text{if } p < q \leq 1-p \\ 1 - \frac{1-q}{p}, & \text{if } 1-p < q \end{cases}$$

$$r(q) = \begin{cases} 1-p, & \text{if } p < q \leq 1-p \\ p, & \text{otherwise.} \end{cases}$$

In this special case, the desired conclusion is

$$\forall q \ E^*(q) \leq \frac{q}{p} + \frac{1-q}{1-p},$$

and the two premises we must establish are

$$\exists E \ \forall q \ E(q) \leq \frac{q}{p} + \frac{1-q}{1-p}, \quad (8)$$

$$\begin{aligned} \forall E \ (\forall q \ E(q) \leq \frac{q}{p} + \frac{1-q}{1-p}) \\ \Rightarrow \forall q \ \tau(E)(q) \leq \frac{q}{p} + \frac{1-q}{1-p}. \end{aligned} \quad (9)$$

The premise (8) is trivial; for example, take $E = \lambda q.0$.

For (9), let E be arbitrary. We wish to show that

$$\begin{aligned} \forall q \ E(q) &\leq \frac{q}{p} + \frac{1-q}{1-p} \\ \Rightarrow \forall q \ \tau(E)(q) &\leq \frac{q}{p} + \frac{1-q}{1-p}. \end{aligned} \quad (10)$$

Picking q arbitrarily on the right-hand side and then specializing the left-hand side at $f_1(q)$, it suffices to show

$$\begin{aligned} E(f_1(q)) &\leq \frac{f_1(q)}{p} + \frac{1-f_1(q)}{1-p} \\ \Rightarrow \tau(E)(q) &\leq \frac{q}{p} + \frac{1-q}{1-p}. \end{aligned} \quad (11)$$

Substituting the definition of τ , we need to show

$$\begin{aligned} E(f_1(q)) &\leq \frac{f_1(q)}{p} + \frac{1-f_1(q)}{1-p} \\ \Rightarrow 1 + r(q)E(f_1(q)) &\leq \frac{q}{p} + \frac{1-q}{1-p}. \end{aligned} \quad (12)$$

The proof breaks into three cases, depending on whether $q \leq p$, $p < q \leq 1-p$, or $q > 1-p$. In the first case, $f_1(q) = q/p$ and $r(q) = p$. Then (12) becomes

$$E\left(\frac{q}{p}\right) \leq \frac{q}{p^2} + \frac{1-\frac{q}{p}}{1-p} \Rightarrow 1 + pE\left(\frac{q}{p}\right) \leq \frac{q}{p} + \frac{1-q}{1-p}.$$

But

$$1 + pE\left(\frac{q}{p}\right) \leq 1 + p\left(\frac{q}{p^2} + \frac{1-\frac{q}{p}}{1-p}\right) = \frac{q}{p} + \frac{1-q}{1-p}.$$

The remaining two cases are equally straightforward. The last case, $q > 1-p$, uses the fact that $p \leq 1/2$.

One can also prove closed properties of more than one function E . For example, as promised, we can show that $E_3^* \leq E_1^*$ whenever $\max((1-p)^2, 1-(1-p)^2) \leq c \leq 1-p$. For this application, B is the space of pairs (E, E') , where E and E' are bounded complex-valued functions on the unit interval, $a = (\lambda x.1, \lambda x.1)$, and $R : B \rightarrow B$ is the bounded linear operator

$$R(E, E') = (\lambda q.r(q)E(f_3(q)), \lambda q.r(q)E'(f_1(q)))$$

with spectral radius $1-p$. The closed property of interest is $E \leq E'$, but we need the stronger induction hypothesis

$$\begin{aligned} \varphi(E, E') \\ = \forall q \ E(q) \leq E'(q) \end{aligned} \quad (13)$$

$$\wedge E'(q) \geq \frac{1}{1-p} \quad (14)$$

$$\wedge p < q \leq 1-p \Rightarrow E'(q) \geq 2 \quad (15)$$

$$\wedge E(q) \leq \frac{q}{p} + \frac{1-q}{1-p} \quad (16)$$

$$\wedge 0 < q \leq p \Rightarrow E(q) = E(q + 1 - p). \quad (17)$$

There certainly exist (E, E') satisfying φ . We have also already argued that induction hypothesis (16) is preserved by τ . The argument for (14) is similar. For (17), if $0 < q \leq p$, then

$$1 - p < q + 1 - p \leq 1,$$

therefore

$$\begin{aligned} r(q) &= r(q + 1 - p) = p \\ f_3(q) &= q/p \\ f_3(q + 1 - p) &= 1 - (1 - (q + 1 - p))/p \\ &= q/p. \end{aligned}$$

It follows that

$$\begin{aligned} 1 + r(q)E(f_3(q)) &= 1 + r(q + 1 - p)E(f_3(q + 1 - p)) \\ &= 1 + pE(q/p). \end{aligned}$$

For (15), if $p < q \leq 1 - p$, then

$$\begin{aligned} r(q) &= 1 - p \\ E'(f_1(q)) &\geq \frac{1}{1 - p} \end{aligned}$$

by the induction hypothesis (14), thus

$$\begin{aligned} 1 + r(q)E'(f_1(q)) &\geq 1 + (1 - p)\frac{1}{1 - p} \\ &= 2. \end{aligned}$$

Finally, for (13), we wish to show

$$1 + r(q)E(f_3(q)) \leq 1 + r(q)E'(f_1(q)),$$

or equivalently,

$$E(f_3(q)) \leq E'(f_1(q)). \quad (18)$$

Since f_1 and f_3 coincide except in the range $c < q \leq 1 - p$, we need only show (18) for q in this range.

It follows from the assumptions in effect that

$$\begin{aligned} p < f_1(q) &= 1 - \frac{1 - q}{1 - p} \\ &\leq 1 - p < f_3(q) = \frac{q}{1 - p}, \end{aligned}$$

thus

$$\begin{aligned} E(f_3(q)) &= E\left(\frac{q}{1 - p} - (1 - p)\right) && \text{by (17)} \\ &\leq \frac{\frac{q}{1 - p} - (1 - p)}{p} + \frac{1 - (\frac{q}{1 - p} - (1 - p))}{1 - p} && \text{by (16)} \\ &= \left(\frac{q}{1 - p} - 1\right)\frac{1 - 2p}{p(1 - p)} + 2 \\ &\leq 2 && \text{since } p, q \leq 1 - p \\ &\leq E'(f_1(q)) && \text{by (15).} \end{aligned}$$

Note that nowhere in this proof did we use any analytic arguments. All the necessary analysis is encapsulated in the proof of Theorem 3.1.

4 Unbounded Solutions

That these coinductive proofs have no basis is reflected in the fact that there exist unbounded solutions in addition to the unique bounded solutions. All unbounded solutions are necessarily noncontinuous, because any continuous solution on a closed interval is bounded.

Theorem 3.1 does not mention these unbounded solutions, because they live outside the Banach space B . Nevertheless, it is possible to construct unbounded solutions to any of the above recurrences. Let G be the graph with vertices $q \in [0, 1]$ and edges $(q, f(q))$. Note that every vertex in G has outdegree 1. Let C be an undirected connected component of G . One can show easily that the following are equivalent:

(i) C contains an undirected cycle;

(ii) C contains a directed cycle;

(iii) for some $q \in C$ and $k \geq 0$, $f^k(q) = q$.

Call C *rational* if these conditions hold of C , *irrational* otherwise. For example, the connected components of 0 and 1 are rational, since $f(0) = 0$ and $f(1) = 1$. There are other rational components besides these; for example, if $p = 1/2$, the component of $q = 2/3$ is rational, since $f^2(2/3) = 2/3$.

Now any solution E must agree with the unique bounded solution E^* on the rational components. This is because if $f^k(q) = q$, then the set $\{f^k(q) \mid k \geq 0\}$ is finite, hence E is bounded on this set, and one can show by an extension of the uniqueness argument above that E and E^* must agree on this set. But the values of E on an entire connected component are uniquely determined by its value on a single element of the component, since $E(q)$ uniquely determines $E(f(q))$ and vice-versa. Thus E and E^* must agree on the entire component.

For an irrational component, since there are no cycles, it is connected as a tree. We can freely assign an arbitrary value to an arbitrarily chosen element q of the component, then extend the function to the entire component uniquely and without conflict.

It therefore remains to show that there exists an irrational component. This follows from the fact that if $f^k(q) = q$, then q is a rational function of p . To see this, note that any $f^k(q)$ is of the form

$$\frac{q}{p^m(1 - p)^{k-m}} - r$$

for some $0 \leq m \leq k$ and $r \in \mathbb{Q}(p)$. This can be shown by induction on k . Solving $f^k(q) = q$ for q gives

$$q = \frac{rp^m(1-p)^{k-m}}{1-p^m(1-p)^{k-m}} \in \mathbb{Q}(p).$$

Thus the component of any real $q \notin \mathbb{Q}(p)$ is an irrational component. There exist uncountably many such q , since $\mathbb{Q}(p)$ is countable. In fact, there are uncountably many irrational components, since each component is countable, and a countable union of countable sets is countable. Moreover, it can be shown that if q_1 and q_2 are in the same component, then $\mathbb{Q}(p, q_1) = \mathbb{Q}(p, q_2)$. This is because if q_1 and q_2 are in the same component, then $f^{k_1}(q_1) = f^{k_2}(q_2)$ for some $k_1, k_2 \in \mathbb{N}$, so

$$\frac{q_1}{p^{m_1}(1-p)^{k_1-m_1}} - r_1 = \frac{q_2}{p^{m_2}(1-p)^{k_2-m_2}} - r_2,$$

therefore $q_1 \in \mathbb{Q}(p, q_2)$ and $q_2 \in \mathbb{Q}(p, q_1)$.

We have thus characterized all the possible solutions.

5 Future Work

There is great potential in the use of proof principles similar to those of Theorem 3.2 for simplifying arguments involving probabilistic programs, stochastic processes, and dynamical systems. Such rules encapsulate low-level analytic arguments, thereby allowing reasoning about such processes at a higher algebraic or logical level. Applications could be found in complex and functional analysis, the theory of linear operators, spectral decomposition, measure theory and integration, random walks, and fractal analysis, functional programming, and probabilistic logic and semantics.

In the examples above, the operators R were uncountable-state sub-stochastic transition matrices. In probabilistic semantics [2, 8], programs are modeled as measurable kernels $R(x, A)$, which can be interpreted as forward-moving measure transformers or backward-moving measurable function transformers. These are linear operators that are nonexpansive (spectral radius ≤ 1), but not necessarily contractive (spectral radius < 1) due to the possibility of nonhalting. Also, the expectation functions considered above were uniformly bounded, but there are examples of random walks on infinite graphs for which this is not true. It would be nice to find rules to handle these cases.

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